OF THE REGISTRAR
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MADRAS.

THE JOURNAL

OF THE

Indian Mathematical Society.

Vol. VI.] FEBRUARY 1914. [No. I.

PROGRESS REPORT.

The following gentlemen have been elected members of the Society :—

(1) *Mr. S. G. Burrow, B. Sc.*, (London) *A. R. C. Sc.*—Principal and Professor of Physics, Baroda College, Baroda;

(2) *Mr. Shiv Narayan, M. A., B. Sc.* (Punjab), *B. E.* (Union, America), *A.M.I.E.E.*,—Assistant Professor of Electrical Engineering, College of Engineering, Poona;

(3) *Mr. Kalluri Sundara Ramiah B.A., L.T.*—Mathematics Assistant, Town High School, Guntur, (*at concessional rate*).

(4) *Mr. K. S. Srinivasachari, B.A., L.T.*—Teacher, Mahant's High School, Tirupati, (Chittor) (*at concessional rate*).

(5) *Mr. S. V. Venkataraya Sastri M.A., L.T.*—Mathematics Lecturer, Victoria College, Palghat, (*at concessional rate*).

2. The Committee feel pleasure to announce that—

(i) *The Tohoku Mathematical Journal*, edited by Professor T. Hayashi, and published at Tokyo (Japan) will be received in exchange for our Journal;

(ii) *The University of Illinois* will send in exchange for our journal annually, from their Department of Mathematics the Ph. D. theses and other mathematical articles as they are published.

(iii) "*Nature*" will be available for issue every week for the current year, through the courtesy of Mr. Balak Ram, I.C.S.

3. The following books have been received from the Publishers—

(1) *Numerical Trigonometry*—by Messrs. Borchardt & Perrott, 2*s*. 6*d*; London, 1913. (George Bell & Sons).

(2) *A Shorter Algebra*—by Messrs Baker and Bourne, 1913, 2/6. (George Bell & Sons).

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(3) *The Calcutta University Calendar* for **1913**, Part IV.

(4) *Brief Biography* and *Popular Account* of the Unparalleled Discoveries of Prof. Dr. T. J. J. Sly—by W. L. Webb. Lyma, Mass, U. S. A., 1913 (T. P. Nichols & Son Co.).

(5) *Leçons de Mathematiques Gēnērales*—by Prof. L. Zoretti, with a preface by P. Appell, Paris, 1914, 20 frs. (Gauthier—Villars).

(6) *Nature*—Vols. 88, 89, 90, & 91, as presents, from Mr. Balak-Ram, I. C. S.

POONA,
31*st Jan.*, 1914.

D. D. KAPADIA,
Hon. Joint Secretary.

UNIVERSITY LECTURES IN MADRAS.

We are pleased to note that a course of University lectures twenty one in number, was recently delivered in the Presidency College, Madras, by Mr. E H. Neville, Fellow, of Trinity College, Cambridge. The course began on January the 5th and terminated on February 7th.

For the benefit of those who could attend the lectures for only one week, Mr. Neville gave two lectures on 'THE NATURE OF REAL AND COMPLEX NUMBERS' and two lectures complete in themselves on 'DIFFERENTIAL GEOMETRY.' The remainder of the course was taken up by a detailed treatment of 'MOVING AXES AND DIFFERENTIAL GEOMETRY,' a subject which Mr. Neville has made his own.

The course was well attended and drew an audience not only from the south of India, but also from so far north as Aligarh. Mr. Neville will, it is hoped, issue his lectures in book form during the course of the year.

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The Spherical Mean Centre.

By M. T. Naraniengar.

1. Let A_1, A_2 be points on a great circle of a sphere whose distances from an origin O on the great circle are α_1, α_2; and let M be a point determined by the equation $\sin(x-\alpha_1)+\sin(x-\alpha_2)=0$.

Then, evidently, $x=\frac{1}{2}(\alpha_1+\alpha_2)$, so that M is the mid-point of A_1A_2;

and $$\sin\alpha_1+\sin\alpha_2=2\sin x\cos\tfrac{1}{2}A_1A_2 \propto \sin x.$$

The point M is called the *spherical mean centre* of A_1A_2.

2. If A_1, A_2, A_3......be any number of points on a great circle whose distances from O are $\alpha_1, \alpha_2, \alpha_3$......, then their *s. m. c.* is M determined by the equation

$$\sin(x-\alpha_1)+\sin(x-\alpha_2)+\ldots=0,$$

or $$\Sigma\sin(A_1M)=0,$$

so that $$\tan x=\tan OM=\Sigma(\sin\alpha_1)/\Sigma(\cos\alpha_1),$$

and $$\sin\alpha_1+\sin\alpha_2+\sin\alpha_3+\ldots\ldots=\Sigma\sin[x-(x-\alpha_1)]$$
$$=\sin x\Sigma\cos(x-\alpha_1)-\cos x\Sigma\sin(x-\alpha_1)$$
$$=\sin x\Sigma\cos(A_1M)\propto\sin x,$$

since $\Sigma\sin(x-\alpha_1)=0$, and M is fixed relatively to A_1A_2......

3. If A_1, A_2,......be a set of points on a great circle whose *s.m.c.* is M, then

$$\Sigma\cos OA_1\propto\cos OM,$$

where O is any point on the great circle.

For,

$$\Sigma\cos OA_1=\Sigma(\cos\alpha_1)=\Sigma\cos[x-(x-\alpha_1)]$$
$$=\cos x\Sigma\cos(x-\alpha_1),\text{ since }\Sigma\sin(x-\alpha_1)=0,$$
$$=\cos x\Sigma\cos(A_1M)\propto\cos OM.$$

Cor. $\Sigma\cos(OA_1)$ is greatest, when $x=0$; that is when O is at M.

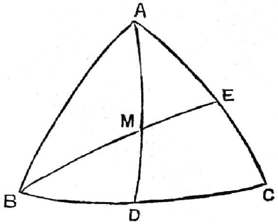
4. Suppose ABC is a spherical triangle and M is a point within the triangle defined by the relation

$$\text{arc }MA+\text{arc }MB+\text{arc }MC=0,$$

where the *plus* sign has to be interpreted as in the case of *vectors*.

Thus to combine arc MB, and arc MC, we may proceed as follows:

Bisect BC at D and join MD. Then the *sum* or *resultant* of arcs MB, MC is X or λ MD, such that sin X $=\lambda$ sin OD, where λ stands for 2 cos $\frac{1}{2}$ BC.



Now, if M be the median centre of a spherical triangle ABC, we know that

$$\sin AM = 2 \cos \tfrac{1}{2} a . \sin MD,$$ (Casey : *Sphl. Trig.* p. 72.) ;

so that
$$\text{arc } MB + \text{arc } MC = X = \text{arc } AM.$$

Thus the point M defined by the equation

$$\text{arc } MA + \text{arc } MB + \text{arc } MC = 0$$

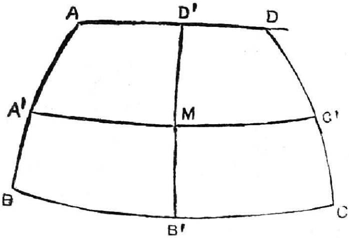
is the centre of medians of the triangle ABC, and may be called the *s. m. c.* of A, B, C.

5. If A, B, C, D......be any number of points on a sphere, the point M defined by the equation

$$\Sigma \, (\text{arc } MA) = 0$$

is the *s. m. c.* of the system, and a similar geometrical interpretation of the equation will hold. Thus in the case of a spherical quadrilateral the *s. m. c.* M is the intersection of the connectors of the middle points of opposite sides, and is such that

$$\text{arc } MA + \text{arc } MB = -(\text{arc } MC + \text{arc } MD).$$



For, if X is the *sum* of arc MA and arc MB,

$$\sin X = 2 \sin MA' . \cos \tfrac{1}{2} AB, \text{ by § 4.}$$

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Now, if P be the pole of B′D′,

$$\cos PA + \cos PB = 2 \cos PA'. \cos \tfrac{1}{2} AB.$$

and
$$\cos PA' = \sin MA'. \sin M.$$

$$\therefore \quad \cos PA + \cos PB = \sin x. \sin M.$$

In other words, $\sin X \sin M = \sin \alpha + \sin \beta$, where α, β are the perpendiculars from A, B on B′D′.

Similarly, the resultant of arcs MC, MD is Y, such that

$$\sin Y \sin M = \text{sum of sines of perpendiculars from C, D.}$$

Hence
$$\sin X = -\sin Y, \quad \therefore X = -Y.$$

Cor. 1. In a spherical quadrilateral, if A′B′C′D′ be the mid-points of AB, BC, CD, DA, and M be the *s. m. c.*

$$\sin MA'. \cos \tfrac{1}{2} AB = \sin MC' \cos \tfrac{1}{2} CD,$$
$$\sin MB'. \cos \tfrac{1}{2} BC = \sin MD' \cos \tfrac{1}{2} AD;$$

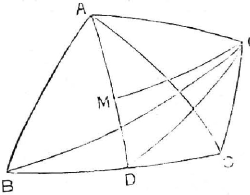
6. Generally, the resultant X of p. OA and q. OB may be defined as follows:

Divide, the arc AB at C into parts whose sines are inversely as $p : q$, then X is represented by λ.OC, such that

$$\sin X = \lambda. \sin OC = (p \cos AC + q \cos CB). \sin OC.$$

7. We can now proceed to give a geometrical construction for the resultant of arcs OA, OB, OC where O is any point whatever.

By § 4 the resultant of arcs OB, OC is $2 \cos \frac{1}{2} a \sin OD$, and the resultant of this and OA is λ OM, where M divides AD inversely as $1 : 2 \cos \frac{1}{2} a$ by § 6.



Hence, the resultant of arc OA, OB, OC, is X or $\lambda \cdot$OM such that

$$\sin X = \lambda \cdot \sin OM = \sin OM \; (2 \cos \tfrac{1}{2} a \cos MD + \cos MA)$$
$$= \sin OM. \; (\cos MA + \cos MB + \cos MC).$$

Similarly, the resultant of arcs OA, OB, OC, OD is X or λOM, such that

$$\sin X = \lambda \sin OM = \sin OM \; \Sigma \cos MA,$$

where M is the *s. m. c.* of ABCD; and so on.

8. Next, let M be a point defined by the relation

$$p\text{MA} + q\text{MB} + r\text{MC} = 0.$$

In this case it is readily seen that the point M is the point of concourse of the arcs AD, BE, CF, where, D, E, F divide the sides of ABC into parts whose sines are in the inverse ratios $p : q : r$.

This point M is the *spherical mean centre* of ABC for *multiples* p, q, r.

If O is any other point the resultant of p OA, q OB, r OC is seen to be X or λ OM, such that

$$\sin X = \lambda \sin OM = \sin OM . \Sigma(p \cos MA);$$

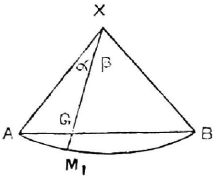
and so on.

9. The *s.m.c.* of a set of points may be obtained from statical considerations as follows:

Let G denote the centroid of masses p, q, r.........placed at A, B, C, ...then the *spherical mean centre* of ABC......... ; for multiples p, q, r......coincides with the projection of G on the spherical surface.

For, the centroid G, of p, q, divides the chord AB into parts inversely as $p : q$; so that its projection divides the arc AB into parts α, β connected by the relation

$$\sin \alpha : \sin \beta = \triangle AXG_1 : \triangle BXG_1$$
$$= AG_1 : BG_1 = q : p.$$



Hence the result is true in the case of A, B.

Similarly, its truth is established in the case of any number of points A, B, C,.........

10. The *s. m. c.* of a set of $(x+y)$ points lies on the arc joining the *s. m. c.* of the first x points to that of the remaining y points.

The proof is obvious statically. Hence, we infer that the several arcs joining the *s. m. c.* of any x points to that of the remaining y points are concurrent.

Cor. 1. The arcs joining the mid-points of opposite connectors of four points are concurrent.

Cor. 2. If in a $\triangle$ ABC D, E, F be the mid-points of the sides and L, M, N those of the arcs joining the ortho-centre to A, B, C, then DL, EM, FN are concurrent ; and the point of concurrence lies on the arc joining the ortho-centre to the median centre.

11. If $x_1, x_2, x_3, \ldots\ldots$ denote the sines of the perpendicular arcs from a set of points $A_1, A_2, A_3, \ldots\ldots$ on any arc X, and x the sine of the perpendicular from M their *s. m. c.* for multiples $m_1, m_2, m_3, \ldots\ldots\ldots\ldots$ then $m.x = \Sigma(m_1 x_1)$, where m denotes $\Sigma(m_1 \cos MA_1)$

This follows from the statical property of the *centroid* of $m_1, m_2 \ldots$ placed at $A_1, A_2, \ldots\ldots\ldots$

For, the *actual* distances of $m_1, m_2, \ldots\ldots$ from the plane of the arc X are proportional to $x_1, x_2 \ldots\ldots$ so that the *actual* distance of the *centroid* from X is $R.\Sigma(m_1 x_1)/\Sigma(m_1)$. Also, the distance of the centroid from the centre of the sphere is d such that

$$d.\Sigma(m_1) = \Sigma(m_1 d_1),$$

where $d_1, d_2, \ldots\ldots$ are the actual distances of $A_1, A_2, \ldots\ldots$ from the plane of the polar circle of M; and $d_1, d_2, \ldots\ldots\ldots$ are equal to $R \cos MA_1$, $R \cos MA_2 \ldots\ldots\ldots$

Thus $$d\Sigma m_1 = R.\Sigma_1 (m_1 \cos MA_1) = m.R.$$

But $$x = \frac{R.\Sigma(m_1 x_1)}{d\Sigma(m_1)}.$$

Hence $$m.x = \Sigma(m_1 x_1).$$

The result may also be deduced trigonometrically from Stewart's Theorem.

Cor. 1. If X is an arc passing through M,

$$\Sigma(m_1 x_1) = 0.$$

Thus the *s. m. c.* may also be defined as the point determined by the above equation.

Cor. 2. The sine of the perpendicular from M on any side $A_r A_{r+1}$ of $A_1 A_2 \ldots\ldots$, varies as the sum of the sines of the perpendiculars from the remaining vertices on it.

Cor. 3. If O be the pole of X

$$m.\cos OM = \Sigma(m_1 \cos OA_1),$$

since $x.\, x_1, x_2 \ldots\ldots$ are equal to $\cos OM$, $\cos OA_1$, $\cos OA_2 \ldots\ldots\ldots$

Cor. 4. If D, E, F be the middle points of the sides of the triangle ABC, the median centre M of ABC is the *s. m. c.* of DEF for multiples $\cos \frac{1}{2} a$, $\cos \frac{1}{2} b$, $\cos \frac{1}{2} c$.

For, we have $\cos OB + \cos OC = 2 \cos \frac{1}{2} a \cos OD$, and two similar results.

$$\therefore \quad \Sigma(\cos OA) = \Sigma(\cos \tfrac{1}{2} a \cos OD).$$

But $$\Sigma \cos OA \propto \cos OM.$$

Thus $\Sigma \cos \frac{1}{2} a \cos OD \propto \cos OM$, and therefore M is the *s m.c.* of D, F, F for multiples $\cos \frac{1}{2} a$, $\cos \frac{1}{2} b$, $\cos \frac{1}{2} c$.

Similarly, if A, B, C, D...be a set of points and X, Y, Z...be the mid-points of AB, BC, CD,..., then the *s. m. c.* of X, Y, Z...for multiples cos $\frac{1}{2}$ AB, cos $\frac{1}{2}$ BC, cos $\frac{1}{2}$ CD......coincides with that of A, B, C, D...for equimultiples.

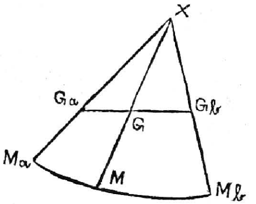
12. If M_a is the *s. m. c.* of a set of x points A_1, A_2......and M_b that of another set of y points B_1, B_2,........., the *s. m. c.* of the combined set divides the arc $M_a M_b$ in the inverse ratio $\Sigma(\cos M_a A_1) : \Sigma(\cos M_b B_1)$.

For, if d_a, d_b, denote the distances of the centroids G_a, G_b of the two sets from the centre of the sphere, the certroid G of the combined set divides $G_a G_b$ in the inverse ratio $x : y$. Also

$$xd_a : y\,d_b = \Sigma(\cos M_a A_1) : \Sigma(\cos M_b B_1), \text{ by § 11 };$$

and

$$\sin M_a M : \sin M_b M = xd_a : ydb.$$



Hence

$$\sin M_a M : \sin M_b M = \Sigma(\cos M_a A_1) : \Sigma(\cos M_b B_1).$$

Cor. 5. If two triangles ABC, A′B′C′ be such that $\cos a + \cos b + \cos c = \cos a' + \cos b' + \cos c'$, then the *s.m.c.* of the six points ABC A′B′C′ bisects the join of the median centres of the triangles.

For, in this case $\Sigma \sin^2 \frac{1}{2} a = \Sigma \sin^2 \frac{1}{2} a'$, and therefore the sum of the squares of the sides of the *chordal* triangles are equal, proving that the centroids of these triangles are equidistant from the centre of the sphere. Hence the result.

13. Let M be the *s. m. c.* of a triangle ABC for multiples n_1, n_2, n_3 and let the radius to M meet the plane of the chordal triangle at G. Then, if x, y, z be the areal coordinates of G, we have

$$x : y : z = n_1 : n_2 : n_3.$$

For, we know that n_1, n_2, n_3 are proportional to the triangular coordinates of M (Casey : § 76). Also x, y, z are proportional to the volumes of the tetrahedra standing on BGC, CGA, AGB and having for their common vertex the centre of the sphere; and these are as $n_1 : n_2 : n_3$. Hence the result.

Similarly if A, B, C, D...be any number of points on a small circle and M be their *s. m. c.* for multiples n_1, n_2, n_3,......the corresponding point

G on the plane of the small circle will have areal coordinates proportional to the Staudtians of the triangles AMB, BMC.........

Cor. 1. The in centre of the chordal triangle projects into the point whose triangular coordinates are proportional to the chords a_1, b_1 c_1, of the chordal triangle. But $a_1 : b_1 : c_1 = \sin\frac{1}{2}a : \sin\frac{1}{2}b : \sin\frac{1}{2}c$. Hence the projection is the point $\sin\frac{1}{2}a : \sin\frac{1}{2}b : \sin\frac{1}{2}c$.

Cor. 2. The orthocentre of the chordal triangle projects into the point whose triangular coordinates are as $\tan A_1 : \tan B_1 : \tan C_1$. But A_1=half the angle subtended by BC at the pole of the circumcircle (Casey: XXXV, 1).

$$\therefore \qquad \tan A_1 \propto \tan\tfrac{1}{2}a \operatorname{cosec}(S-A) \propto \cot(S-A).$$

Hence, the projection is the point $\cot S-A : \cot S-B : \cot S-C$.

14. We have

$$\Sigma(m_1 \cos OA_1) = m \cos OM \text{ by § 11, } Cor. \ 3.$$

Let O coincide with $A_1 A_2$......in succession; then

$$m_1 + m_2 \cos A_1A_2 + m_3 \cos A_1A_3 + \ldots\ldots = m\cos A_1M$$
$$m_1 \cos A_2A_1 + m_2 + m_3 \cos A_2A_3 + \ldots\ldots = m\cos A_2M$$
$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

Multiply these equations by m_1, m_2, m_3..............and add:

$$\Sigma(m_1^2) + 2\Sigma(m_1m_2 \cos A_1A_2) = m\Sigma(m_1 \cos A_1N) = m^2, \text{ by § 11.}$$

In particular, for a triangle ABC, if n_1, n_2, n_3 be the triangular co-ordinates of any point M,

$$\Sigma n_1^2 + 2\Sigma(n_1n_2 \cos c) = n^2.$$

15. For any two sets of points A_1A_2.........A_x; B_1B_2........B_y and multiples m_1, m_2......m_x; n_1, n_2......n_y, to prove that

(i) $\Sigma\{ m_p n_q \cos A_pB_q \} = mn \cos M_a M_b$;

(ii) $\Sigma(m_p n_q \cos OA_p . \cos OB_q) = m\, n \cos OM_a . \cos OM_b$.

We have

$$\left.\begin{array}{l} \Sigma m_p \cos B_1A_p = m \cos B_1 M_a \\ \Sigma m_p \cos B_2A_p = m \cos B_2 M_a, \\ \ldots \qquad \ldots \qquad \ldots \\ \ldots \qquad \ldots \qquad \ldots \end{array}\right\} [p=1, 2, \ldots\ldots x].$$

Multiply these by n_1, n_2......and add; the result is

$$\Sigma(m_p n_q \cos A_p B_q) = m\Sigma n_q \cos B_q M_a, \quad [q=1, 2, 3\ldots\ldots y]$$
$$= m\, n \cos M_aM_b.$$

Again

$$\Sigma m_p \cos OA_p = m \cos OM_a. \quad [p.=1, 2\ldots\ldots x]$$

and $$\Sigma n_q \cos OB_q = n \cos OM_b. \quad [q=1, 2, \ldots\ldots y]$$

Multiplying we get

$$\Sigma(m_p\, n_q \cos OA_p \cdot OB_q) = m.n.\ \cos OM_a \cos OM_b.$$

Cor. 1. The two sets have the same *s. m. c.*, if $M_a\, M_b = 0$, the condition for which is

$$\Sigma(m_p\, n_q \cos A_p\, B_q) = m.n.$$

Cor. 2. If the arc $M_a\, M_b$ is a quadrant, $\cos M_a\, M_b = 0$, and therefore the condition for this is

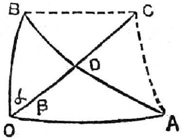
$$\Sigma(m_p\, n_q \cos A_p\, B_q) = 0.$$

Cor. 3. Two spherical triangles ABC, A′B′C′ have a common median centre, if the sum of the cosines of all the arcs joining their vertices is equal to

$$\sqrt{\{3+2(\cos a+\cos b+\cos c)\}\ \{3+2(\cos a'+\cos b'+\cos c')\}}$$

16. Consider the resultant of the set $\Sigma m_p n_q\, A_p B_q$ for different values of p & q. By § 8, the resultant of $n_1 A_1 B_1$, $n_2 A_1 B_2$.........$n_y A_1 B_y$ is represented by $n\ A_1 M_b$, where $n = \Sigma(n_y \cos M_b B_q)$; that of $n\ {}_1A_2B_3$' $n_2 A_2 B_b$.........$n_y.A_2 B_y$ by $n\ A_2 M_b$; &c. Also, the resultant of $m_1 A_1 M_b$, $m_2 A_2 M_b$.........is represented by $m \cdot M_a M_b$ where $m = \Sigma(m_p \cos M_a A_p)$. Hence the resultant of the set $\Sigma m_p n_q A_p B_q$ is $mn\ M_a M_b$,

17. Returning to § 4 let us examine in detail how far the analogy between *plane vectors* and *spherical arcs* can be stretched.



Let OA, OB be two great circle arcs; then their *resultant* is obtained as follows:

Through A, B draw parallels (arcs of small circles) to OB, OA respectively intersecting at C; OC shall be the resultant required. For, denoting sin OA, sin OB, sin OC by x, y, z

$$z \sin \beta = y \sin(\alpha+\beta);\quad z \sin \alpha = x \sin(\alpha+\beta).$$

$$\therefore \qquad x : y : z = \sin \alpha : \sin \beta : \sin(\alpha+\beta).$$

Now $x \sin \beta = y \sin \alpha$ is the condition that D should the middle point of AB; and

$$\begin{aligned} z = x \sin(\alpha+\beta)/\sin \alpha &= \sin AB \cdot \sin B/\sin \alpha \\ &= \sin AB \cdot \sin OD/\sin BD \\ &= 2 \cos \tfrac{1}{2} AB \cdot \sin OD, \end{aligned}$$

so that OC represents the resultant of OA, OB.

Cor. 1. Put $\theta=\alpha+\beta$, then

$$z \sin \beta = y \sin \theta; \quad z \sin \alpha = x \sin \theta.$$

Multiply by $\cos \alpha$, $\cos \beta$ and add :

$$z = x \cos \beta + y \cos \alpha.$$

That is z is the sum of the resolved parts of x and y along the resultant.

Cor. 2. Again $z \cos \beta = x \cos^2 \beta + y \cos \alpha \cos \beta$

$$= x - x \sin^2 \beta + y \cos \alpha \cos \beta.$$

$$= x - y \sin \alpha \sin \beta + y \cos \alpha \cos \beta = x + y \cos \theta.$$

$$\therefore \quad z^2 = x^2 + y^2 + 2\,xy \cos \theta.$$

Cor 3. The sum of the resolved parts of x and y along any arc through O is equal to the resolved part of z along the same.

For $\quad z \sin \beta = y \sin \theta,$

$$z \cos \beta = x + y \cos \theta.$$

$$\therefore \quad z \cos (\beta+\lambda) = z \cos \beta \cos \lambda - z \sin \beta \sin \lambda$$

$$= x \cos \lambda + y (\cos \theta \cos \lambda - \sin \theta \sin \lambda),$$

$$= x \cos \lambda + y \cos (\theta + \lambda).$$

18. The resultant of any number of arcs OA, OB, OC,...is found as in the case of plane vectors by resolving along two perpendicular arcs through O and compounding the components.

Thus if X, Y, denote the sums of components of x, y,...along these rectangular arcs the sine of the resultant is Z, such that

$$Z^2 = X^2 + Y^2.$$

and the inclination of the resultant is $\tan^{-1} (Y/X)$

Cor 1. The conditions that OA, OB,...should have a *null* or *zero* resultant are that $X=0$, and $Y=0$; and conversely, if these conditions are satisfied the resultant is zero ; that is, the point O is the *s. m. c.* of the system of points A, B, C,...

Cor 2. Defining the *moment* of a spherical arc round any point as the product of the sine of the arc into the sine of the perpendicular from the point on the arc, it is easy to see that the moment of the resultant Z is the sum of the moments of the components. Hence follow other conditions for zero resultant, as in Statics.

19. If we replace a set of points A_1, A_2,........by a set of arcs, we have a definite arc called *the spherical mean axis* corresponding to the *s. m. c.* of the points A_1, A_2, whose properties follow from the reciprocal therems relating to poles and polars.

Equi-Brocardian Triangles.

By J. C. Swaminarayan, M.A.

1. Let BC be a given base. On both sides of BC, describe equilateral triangles BPC and BQC. If A is the vertex of any triangle on the base BC such that its Brocard angle is equal to a known angle w, we have from $\triangle$ APB

$$\begin{aligned} AP^2 &= a^2+c^2-2ac\cos(B+60^\circ) \\ &= a^2+c^2-ac\cos B+2ac\sin B\sin 60^\circ \\ &= a^2+c^2-\tfrac{1}{2}(a^2+c^2-b^2)+2\triangle\sqrt{3} \\ &= \tfrac{1}{2}(a^2+b^2+c^2)+2\triangle\sqrt{3} \\ &= 2\triangle(\cot w+\sqrt{3}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1) \end{aligned}$$

Similarly, from $\triangle$AQB

$$AQ^2 = 2\triangle(\cot w-\sqrt{3}) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

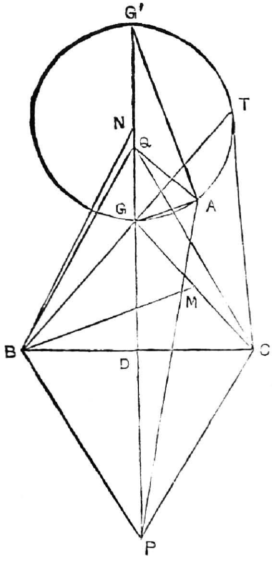
The latter result shows that $\cot w \not< \sqrt{3}$.

2. From this it is apparent that the ratio AP : AQ depends only on the value of the Brocard angle. Because

$$\frac{AP}{AQ} = \left\{\frac{\cot w+\sqrt{3}}{\cot w-\sqrt{3}}\right\}^{\frac{1}{2}}.$$

Hence the locus of the vertex A of the triangle whose Brocard angle is equal to w can be obtained by bisecting $\angle$PAQ internally and externally by lines AG and AG′, cutting PQ in G,G′ respectively. The locus of A will be the circle on GG′ as diameter.

3. Putting, AP$=\alpha$, AQ$=\beta$, we can easily prove that if D is the middle point of BC,



$$DG=PG-PD=\frac{a\sqrt{3}}{2}\left\{\frac{2\alpha}{\alpha+\beta}-1\right\}=\frac{a\sqrt{3}}{2}.\frac{\alpha-\beta}{\alpha+\beta}$$

Similarly
$$DG'=\frac{a\sqrt{3}}{2}.\frac{\alpha+\beta}{\alpha-\beta}.$$

Hence, if N is the centre of the circle on GG′ as diameter,

$$\begin{aligned} DN &= \tfrac{1}{2}(DG+DG') \\ &= \frac{a\sqrt{3}}{4}\left\{\frac{\alpha+\beta}{\alpha-\beta}+\frac{\alpha-\beta}{\alpha+\beta}\right\} \\ &= \frac{a\sqrt{3}}{4}\left\{\frac{2(\alpha^2+\beta^2)}{\alpha^2-\beta^2}\right\} \\ &= \frac{a\sqrt{3}}{2}\left\{\frac{AP^2+AQ^2}{AP^2-AQ^2}\right\} \\ &= \frac{a\sqrt{3}}{2}\left\{\frac{\cot w+\sqrt{3}+\cot w-\sqrt{3}}{\cot w+\sqrt{3}-\cot w+\sqrt{3}}\right\} \\ &= \frac{a}{2}\sqrt{3}.\frac{\cot w}{\sqrt{3}}=\frac{a}{2}\cot w \end{aligned}$$

4. Now

$$\cot B\hat{N}D=\frac{DN}{BD}=\frac{\frac{1}{2}a\cot w}{\frac{1}{2}a}=\cot w.$$

$$\therefore \qquad B\hat{N}D=w.$$

Hence BC subtends at N an angle equal to $2w$.

5. Taking BC and DN as the axes of x and y, we obtain the equation of the locus of A in the form

$$\begin{aligned} x^2+(y-\tfrac{1}{2}a\cot w)^2 &= GN^2 \\ &= \{\tfrac{1}{2}a\cot w-DG\}^2 \end{aligned}$$

But
$$\begin{aligned} DG &= a\frac{\sqrt{3}}{2}\frac{\alpha-\beta}{\alpha+\beta} \\ &= a\frac{\sqrt{3}}{2}.\frac{\sqrt{\cot w+\cot 30^\circ}-\sqrt{\cot w-\cot 30^\circ}}{\sqrt{\cot w+\cot 30^\circ}+\sqrt{\cot w-\cot 30^\circ}} \\ &= \frac{a}{2}\cot w+\frac{a}{2}\sqrt{\cot^2 w-3}. \end{aligned}$$

$$\therefore \qquad x^2+(y-\tfrac{1}{2}a\cot w)^2=\tfrac{1}{4}a^2\cot^2 w-\tfrac{3}{4}a^2.$$

$$\therefore \qquad x^2+y^2-ay\cot w+\tfrac{3}{4}a^2=0. \qquad \ldots \qquad \ldots \qquad (3)$$

The circle (3) is called Neuberg's circle, after the distinguished geometer who first studied its properties.

6. It is interesting to notice that

$$\angle NBD=90^\circ-w,$$
$$\angle NBQ=30^\circ-w,$$
and
$$\angle NBP=150^\circ-w.$$

$$\therefore \quad \frac{NQ}{NP}=\frac{\triangle NBQ}{\triangle NBP}=\frac{\frac{1}{2}NB\cdot BQ\sin(30^\circ-\omega)}{\frac{1}{2}NB\cdot AP\sin(150^\circ-\omega)}$$

$$=\frac{\sin 30^\circ \sin\omega(\cot w-\cot 30^\circ)}{\sin 150^\circ \sin w(\cot w-\cot 150^\circ)}$$

$$=\frac{\cot w-\cot 30^\circ}{\cot w+\cot 30^\circ}=\frac{AQ^2}{AP^2}.$$

Also $\quad NQ\cdot NP=\frac{1}{2}a(\cot w-\cot 30^\circ).\frac{1}{2}a(\cot w+\cot 30^\circ)$

$=\frac{1}{4}a^2(\cot^2 w-3)$

$=$square of the radius of Neuberg's circle from (3),

$=NA^2.$

Hence NA touches the circum-circle of $\triangle APQ$ and $\angle NAQ=\angle QPA$.

It is obvious that Neuberg's circle cuts the circum-circle of $\triangle QAP$ orthogonally.

7. As G and G′ lies on Neuberg's circle, the Brocard angles of the triangles BGC and BG′C will each be equal to w. In the triangle BG′C, draw BM the median through B. Then

$$\cot G\hat{B}M=\frac{2BG^2+2BM^2-\frac{1}{2}GC^2}{4\triangle GBC}$$

$$=\frac{2BG^2+BG^2+BC^2-\frac{1}{2}GC^2-\frac{1}{2}GC^2}{4\triangle BGC}$$

$$=\frac{BG^2+CG^2+BC^2}{4\triangle BGC}, \text{ since } BG=CG,$$

$$=\cot w.$$

$$\therefore \quad GBM=w.$$

If $\angle BGC$ is denoted by 2λ, then

$$\frac{GB}{GM}=\frac{2}{1}=\frac{\sin(2\lambda+w)}{\sin w}$$

$$\therefore \quad \sin(2\lambda+w)=2\sin w \quad \ldots \quad \ldots \quad (4)$$

Similarly, if $\angle BG'C=2\lambda'$,

$$\sin(2\lambda'+w)=2\sin w.$$

Thus $\angle BGC$ and $\angle BG'C$ are determined by the equation

$$\sin(2V+w)=2\sin w,$$

the greater value of V corresponding to $\angle BGC$ and the smaller to $\angle BG'C$.

In fact, BGC and BG′C are the vertical angles of two isosceles triangles on the base BC, having their Brocard angles equal to w. It is a well-known fact that Steiner-angles satisfy the equation

$$\sin(2V+w)=2\sin w.$$

Hence the Steiner angles may be defined as the vertical angles of two isosceles triangles on a given base such that their Brocard angles are equal to a given angle less than 30°.

8. Casey, defines Steiner angles as the angles which the tangents from B or C to Neuberg's circle make with BC. We can easily prove that the angles which these tangents make with BC are equal to the angles BGC and BG′C. Before doing so, we must notice the important fact that the length of the tangent from B or C to Neuberg's circle (3) is equal to $\sqrt{\frac{a^2}{4}+\frac{3a^2}{4}}$ or a.

Let BG be produced to cut Neuberg's circle again in T; then

$$BG \cdot BT = BC^2,$$

and $$\angle GBC = \angle GCB = \angle BTB.$$

$$\therefore \quad BC = CT.$$

Hence CT is a tangent to Nenberg's circle and

$$\triangle BGC \text{ is similar to } \triangle BCT.$$

$$\therefore \quad \angle BGC = \angle BCT.$$

Similarly, if CT′ is the other tangent from C, BT′ passes through G′, and $\angle BG'C = \angle BCT'$.

Thus our definition of Steiner angles agrees with Casey's definition.

SHORT NOTES.

The General Conic in Trilinears.

1. *The discrimination of the conics given by the general equation.*

Let $u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0$ be the conic, which will be denoted by $(u, v, w, u', v' w')(\alpha, \beta, \gamma)^2=0$.

By multiplication of determinants we easily prove that when a conic is transformed by any linear substitution

$$X=l_1x+m_1y+n_1z$$
$$Y=l_2x+m_2y+n_2z$$
$$Z=l_3x+m_3y+n_3z$$

the discriminant of the transformed equation is equal to the discriminant of the original equation multiplied by the square of the determinant formed by the l, m, n's. Askwith: *Analytical Geometry*, § 353; Bromwich, *Quadratic Forms.* We have here

$$x=\alpha x_1+\beta x_2+\gamma x_3,$$
$$y=\alpha y_1+\beta y_2+\gamma y_3,$$
$$z=1=\alpha+\beta+\gamma;$$

where (x, y) are the rectangular coordinates of the point (α,β,γ); $(x_1 y_1)$... are the vertices of the triangle of reference, and the determinant of the substitution is $2\triangle$.

If, therefore $(a b c f g h)(x, y, 1)^2$ transforms to $(u, v, w, u', v', w')(\alpha\beta\gamma)^2$, we have

$$\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} = 4\triangle^2 \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

Also $\quad (a, b, c, f, g, h)(x, y, 1)^2+\lambda$

transforms to $\quad (u, v, w, u', v', w')(\alpha, \beta, \gamma)^2+\lambda(\alpha+\beta+\gamma)^2$,

or $\quad (u+\lambda, v+\lambda, w+\lambda, u'+\lambda, v'+\lambda, w'+\lambda)(\alpha, \beta, \gamma)^2$

Hence

$$4\triangle^2 \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c+\lambda \end{vmatrix} = \begin{vmatrix} u+\lambda, & w'+\lambda, & v'+\lambda \\ w'+\lambda, & v+\lambda, & u'+\lambda \\ v'+\lambda, & u'+\lambda, & w+\lambda \end{vmatrix} = \begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ -\lambda, & -\lambda, & -\lambda, & 1 \end{vmatrix}$$

Equating co-efficients of λ

$$4\Delta^2\,(ab-h)^2=-\begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}=-\sigma,\text{ say.}$$

Thus the conic is an ellipse, parabola, or hyperbola according as σ is negative, zero or positive.

2. *To obtain the axes of the general conic.*

When referred to its centre $(a,b,c,f,g,h)\,(x,y,1)^2=0$ becomes

$$ax^2+2\,hxy+by^2+\frac{D}{ab-h^2}=0,\text{ where } D=\begin{vmatrix} a, h, g \\ h, b, f \\ g, f, c \end{vmatrix}$$

and the magnitudes of the axes are given by

$$\left(\frac{1}{r^2}+\frac{a}{\lambda}\right)\left(\frac{1}{r^2}+\frac{b}{\lambda}\right)=\frac{h^2}{\lambda^2},\text{ where } \lambda=\frac{D}{b-h^2}.$$

We have already found D and $ab-h^2$ and it remains to find $a+b$.

Now $(a,b,c,f,g,h)(x,y,1)^2+k(x^2+y^2)$ transforms to

$$(u,v,w,u',v',w')(\alpha,\beta,)^2+k(x\gamma_1{}^2+y\gamma_1{}^2,\ldots,\ldots,x_2x_3+y_2y_3,\ldots,\ldots)(\alpha\beta\gamma)^2.$$

Therefore from § 1,

$$4\Delta^2\begin{vmatrix} a+k, & h \\ h, & b+k \end{vmatrix}=-\begin{vmatrix} u+ku_1, & w'+kw_1', & v'+kv_1', & 1 \\ w'+kw_1', & v+kv_1, & u'+ku_1', & 1 \\ v'+kv_1', & u'+ku_1', & w+kw_1, & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

where $u_1,\ldots\ldots u_1'$, are written for $x_1^2+y_1^2,\ \ldots,x_2x_3+y_2y_3,\ldots$

Now
$$-\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}=U+V+W+2U'+2V'+2W'$$
$$=(uv+uw+vw-u'^2-v'^2-w'^2-2uu'-2vv'-2ww'+2u'v'+2u'w'+2v'w')$$
$$=f(u,u')\text{ say.}$$

Equating the co-efficients of k on both sides

$$4\Delta^2(a+b)=\Sigma u\frac{\partial f(u_1,\,u_1')}{\partial u_1}+\Sigma u'\frac{\partial f(u_1,\,u'_1\theta)}{\partial u_1'}$$
$$=\Sigma u(v_1+w_1-2u_1')+\Sigma u'(-2u_1'-2u_1+2v_1'+2w_1')$$
$$=\Sigma u(x_2^2+y_2^2+x_3^2+y_3^2-2(x_2x_3+y_2y_3))$$
$$-2\Sigma u'(x_2x_3+y_2y_3+x_1^2+y_1^2-x_1x_3-y_1y_3-x_1x_2-y_1y_2)$$
$$=\Sigma ua^2-2\Sigma u'bc\cos A,$$

since $\qquad a^2=(x_2-x_3)^2+(y_2-y_3)^2$, and

$2bc\cos A=(x_3-x_1)^2+(y_3-y_1)^2+(x_1-x_2)^2+(y^2-y_2)_3-)x_2-x_3)^2-y)_2-y_3)^2$

$=$co-efficient of u' written above.

This of course is the condition for a rectangular hyperbola.

The equation giving the axes is

$$\frac{\lambda^2}{r^4}+(a+b)\frac{\lambda}{r^2}+ab-h^2=0.$$

$$\text{Writing } \rho=\frac{\lambda}{r^2}=\frac{D}{ab-h^2}\cdot\frac{1}{r^2}=-\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} \div r^2 \begin{vmatrix} u, & w,' & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}$$

the axes are given by

$$4\,\Delta^2\rho^2+\rho(\Sigma ua^2-2\Sigma u'bc\cos A)-\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}=0.$$

3. *The conditions for a circle.*

The circumcircle being assumed to be $a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta=0$, for a circle, the expression

$$(u,\ v,\ w,\ u',\ v',\ w')\ \ (\alpha\beta\gamma)^2+\lambda\ \{\ a^2\beta\gamma-b^2\gamma(\beta+\gamma)-c^2\beta(\beta+\gamma)\ \}$$

must be divisible by $\alpha+\beta+\gamma$ and must therefore be identical with

$$(\alpha+\beta+\gamma)\ \{\ u\alpha+(v-\lambda c^2)\beta+(w-\lambda b^2)\gamma\ \}.$$

Equating co-efficients of $\beta\gamma$, we get

$$2u'-\lambda(b^2+c^2-a^2)=v+w-\lambda(b^2+c^2).$$

Hence $\qquad \lambda=\dfrac{v+w-2u'}{a^2}=\dfrac{u+w-2v'}{b^2}=\dfrac{u+v-2w'}{c^2},$

by symmetrical considerations.

A. C. L. Wilkinson.

On Tangential Equations.

The following article is the development of a method indicated in R. A. Roberts' *Examples on Conics* (1884).

Let $a^2l^2+b^2m^2=1$ be the tangential equation of an ellipse referred to its axes and $(Xl+Ym-1)^2-R^2(l^2+m^2)=0$, that of a circle [centre (X,Y,) radius R] ; then

$$a^2l^2+b^2m^2-1+\frac{\lambda^2}{l^2}\{\ (Xl+Ym-1)^2-R^2(l+m^2)\ \}=0, \qquad \ldots \quad (1)$$

denotes a cotangential conic. (*vide*: p. 30, Ex. 44)

When (1) splits up into factors, it represent two points. The discriminant is

$$\frac{X^2}{a^2-\lambda^2}+\frac{Y^2}{b^2-\lambda^2}+\frac{R^2}{\lambda^2}-1 =0.$$

For the values of λ derived from this equation the two points are the extremities of a diagonal of the tangential quadrilateral. At the same time, for these particular values of λ, the points as above determined are the intersections of

$$(a^2-\lambda)^2 l^2+(b^2-\lambda^2)m^2-1=0, \text{ and } Xl+Ym-1=0;$$

i.e., of $$\frac{x^2}{a^2-\lambda^2}+\frac{y^2}{b^2-\lambda^2}-1 =0, \text{ and } \frac{xX}{a^2-\lambda^2}+\frac{yY}{b^2-\lambda^2}=1.$$

Hence, the opposite vertices of the cotangential quadrilateral of a circle and a conic lie on a confocal, being the intersections of the confocal with the polar of the centre of the circle with respect to that confocal. ... ... (2).

Hence, also, if λ_1, λ_2, λ_3 are the three roots of the equation $\frac{X^2}{a^2-\lambda^2}=\frac{Y^2}{b^2-\lambda^2}+\frac{R^2}{\lambda^2}=1$, we have, as in Lamb's *Hydrodynamics*, §. 112,

$$X^2=\frac{(a^2-\lambda_1^2)(a^2-\lambda_2^2)(a^2-\lambda_3^2)}{a^2(a^2-b^2)}, \quad Y^2=\frac{(b^2-\lambda_1^2)(b^2-\lambda_2^2)(b^2-\lambda_3^2)}{b^2(a^2-b^2)},$$

$$R^2=\frac{\lambda_1^2\lambda_2^2\lambda_3^2}{a^2b^2},$$ results worked out in Roberts (p. 31).

Thus $$X=\frac{a_1a_2a_3}{ac}, \quad Y=\frac{\sqrt{(c^2-a_1^2)(c^2-a_2^2)(c^2-a_3^2)}}{bc},$$

$$R=\frac{\sqrt{(a^2-a_1^2)(a^2-a_2^2)(a^2-a_3^2)}}{ab}, \qquad \text{... (3).}$$

where $a_1a_2a_3$ are the sami major axes of the three confocals through the three pairs of vertices.

For the incircle the confocals are hyperbolasand for the ex-circle two of the confocals are ellipses. Again, if s be the semi-perimeter of the triangle formed by three tangents to the in circle and the ellipse, we have $Rs^2=R_1R_2R_3$ where R_1, R_2, R_3, are the ex-radii and R the in-radius. On substituting for R_1, R_3, R_3 and R from (3) and remembering the distinction between *in*- and *ex*-circles as above referred to

$$s=\frac{\sqrt{-(a^3-\alpha_1^2)(a^2-\alpha_2^2)(a^2-\alpha_3^3)}}{ab},$$

α_1, α_2, α_3 being the semi-axes of the confocal ellipses through the three vertices. ... ... (4).

From (3) we see that, if two of the vertices of a triangle move along confocal hyperbolas, the incentre is given by

$$acx = a_1 a_2 a_3,\ b^2c^2y^2 = (c^2 - a_1^2)(c^2 - a_2^2)(c^2 - a_3^2)$$

whence, eliminating a_3, the incentre moves on the ellipse

$$\frac{a^2x^2}{a_1^2a_2^2} + \frac{b^2y^2}{(c^2 - a_1^2)(c^2 - a_2^2)} = 1,$$

a result given, but not worked out, by Roberts (p. 34, Ex. 46).

Also, when two of the vertices of a triangle move along confocal ellipses, one of the ex-centres lies on a fixed coaxial conic, as seen above and the third corner moves on a confocal ellipse, so that, from (4), $2s$ or the perimeter of the triangle is constant. In the case of a triangle of maximum perimeter the sides touch a confocal, so that its perimeter is $(a^2 - a_1^2)/a_1 b_1$ from (4). The invariant relation is

$$\frac{a_1}{a} + \frac{b_1}{b} = 1, \text{ or } \left(\frac{a_1 - a}{a}\right)^2 = \frac{b_1^2}{b^2} = \frac{a_1^2 - c^2}{b^2}.$$

$$\therefore \quad -b^2(a_1^2 + a^2 - 2a_1a) + a^2(a_1^2 - c^2) = 0.$$

$$\therefore \quad a_1^2c^2 - 2a_1ab^2 - a^4 = 0.$$

$$\therefore \quad a_1 = \frac{ab^2 \pm \sqrt{a^2b^4 + a^4c^2}}{c^2} = \frac{ab^2 \pm a\sqrt{a^4 - a^2b^2 + b^4}}{c^2}$$

For obvious reasons we neglect the plus sign as giving a value greater than a, so that

$$a^2 - a_1^2 = \frac{a^2}{c^4}\{(a^2 - b^2)^2 - b^4 - (a^4 - a^2b^2 + b^4) + 2b^2\sqrt{a^4 - a_2^2b^2 + b^4}\}$$

$$= \frac{a^2b^2}{c^4}\{2\sqrt{a^2 - a^2b^2 + b^4} - (a^2 + b^2)\}$$

$$= \frac{3a^2b^2}{(a^2 + b^2) + 2\sqrt{a^4 - a^2b^2 + b^4}}.$$

Thus remembering that $b_1^2 = a_1^2 - c^2$, we have for the semi-perimeter

$$s = \frac{\sqrt{3}\{a^2 + b^2 + \sqrt{a^4 - a^2b^2 + b^4}\}}{(a^2 + b^2 + 2\sqrt{a^4 - a^2b^2 + b^4})^{\frac{1}{2}}}$$

a result given in Wolstenholme, Problem 1060. ... ... (5),

Again taking the circle of curvature at a point P and the conic, three of the sides of the quadrilateral are coincident along the tangent at P, and the other common tangent of this circle and the ellipse cuts it in a point Q, such that P and Q lie on the same con-focal through P. ... ... (6).

Further, if ABC be a triangle inscribed in a conic and circumscribed to a confocal, D the point of contact of the latter with BC and E the common point of BC and the fourth common tangent of the in circle of

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ABC and the inner confocal, it follows from (2) that A and E lie on on the same confocal through A. But, by a well-known theorem, A and D also lie on the same confocal. Hence E is the point of intersection of BC with the confocal through A other than D. But D and the common point of the common tangents of the osculating circle at D and the inner conic also lie on the same confocal. It follows therefore that the fourth common tangent of the incircle and in-conic of ABC passing through E, touches the osculating circle at D. Hence the fourth common tangent of the incircle and the in-conic of ABC touches the osculating circle at D.

Similarly, for the points of contact with the other sides. Hence the proposition that, if a triangle be inscribed in a conic and circumscribed to a confocal, the osculating circle at the point of contact of the sides of the triangle with the latter are all touched by the fourth common tangent of the inconic and the incircle of the triangle—a proposition otherwise proved in Roberts (p. 113, Ex. 193) ... ... (7).

Further, if the in circle of a self-conjugate triangle, with reference to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, be $(x-\alpha)^2+(y-\beta)^2=\lambda^2$, forming the invariant condition $\Theta'=0$ for the circumscription of a self-conjugate triangle, we have

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1-r^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=0,$$

which may be written
$$\frac{\alpha^2}{a_1^2-h^2}+\frac{\beta^2}{b_1^2-h^2}+\frac{r^2}{h^2}-1=0,$$

where
$$a_1^2=\frac{a^4}{a^2+b^2},\quad b_1^2=\frac{b^4}{a^2+b^2},\quad h^2=-\frac{a^2b^2}{a^2+b^2}.$$

On comparison with results (3), (4), we see that the intersection of the polar of (α,β) with respect to $\frac{x^2}{a_1^2-h^2}+\frac{y^2}{b_1^2-h^2}=1$, or $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with the same determine two points the tangents from which to $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$. are also tangents to $(x-\alpha)^2+(y-\beta)^2=r^2$.

Hence it follows that if (α,β) lie on the director circle of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, since it polar with reference to the same touches $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$ at the foot of the $\perp r$ from (α,β) on the polar, the in-

circle touches the polar at the same point as the confocal; and the envelope of the circle is the confocal

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2},\qquad \text{(Roberts, p. [illegible], Ex. 67).}$$

A. N. RAGHAVACHAR.

The Face of the Sky for March and April 1914.

Sidereal time at 8 p.m.

	March.				*April.*		
D.	H.	M.	S.		H.	M.	S.
1	6	34	12	...	8	37	22
8	7	1	48	...	9	4	58
15	7	29	23	...	9	32	34
22	7	56	59	...	10	0	9
29	8	24	45	...	10	27	45

From this table the constellations visible during the evenings of March and April can be ascertained by a reference to the positions as given in a star-atlas.

The Sun

enters the vernal equinox on March 21 at 4·41 P.M.

Phases of the Moon.

	March.					*April.*			
	D.	H.	M.			D.	H.	M.	
First Quarter ...	5	10	33	A.M.	...	4	1	11	A.M.
Full Moon ...	12	9	48	„	...	10	6	58	P.M.
Last Quarter ...	19	1	9	„	...	17	1	22	„
New Moon ...	26	11	39	„	...	25	4	52	„

Eclipses.

The eclipses of February and March are invisible at Madras.

Planets.

Mercury which was in superior conjunction with the sun on January 22, attains its greatest (E) elongation on February 22, is stationary on

March 1, is in inferior conjunction with the sun on March 10, is again stationary on March 23. It attains its greatest (W) elongation on April 7. It is in conjunction with the moon on February 26, March 24, and April 23.

Venus is in superior conjunction with the sun on February 12, and is an evening star after that date. It is in conjunction with the moon on March 28.

Mars which was in opposition to the sun on January 5, is stationary on February 13. It is in conjunction with the moon on March 7 and April 4, and with Neptune on April 21.

Jupiter which was in conjunction with the sun on January 20, is a morning star. It is in conjunction with the moon on February 22 and on April 19 at 5-43 A.M. and with Uranus on March 4.

Saturn is in quadrature to the sun on March 3. It is in conjunction with the moon on March 5, April 1 at 6 P.M. and on April 29.

Uranus was in conjunction with the sun on January 29 ; and will be so with the moon on February 23, March 22, and on April 18 at 4-20 A.M.

Neptune which was in opposition to the sun on January 18, becomes stationary on April 6. It is in conjunction with the moon on February 9, and on March 8 at 1-36 A.M.

V. RAMESAM.

SOLUTIONS.

Question 366.

(M. KANNAN, B.A., L.T.) :—In a triangle ABC, PP′is a diameter of the circumcircle, and the Simson lines of P, P′ interest at ω. Shew that the length of the perpendicular from ω on PP′ is 2R cos α cos β cos γ, where α, β, γ are the inclinations of PP′ to the sides of the triangle.

Additional solution by 'Zero' and M. Satyanarayana.

Take P as the origin of polar coordinates and PP′ as the initial line. If the vectorial angles of A, B, C be x, y, z the Simson-line of P is

$$r\cos(\theta - x - y - z) = 2\text{R}\cos x\cos y\cos z.$$

Hence the coordinates of the foot of the perpendicular from P on the Simson-line are

$$2\text{ R}\cos x\cos y\cos z,\ (x+y+z).$$

When referred to the centre as the origin, the coordinates are therefore

$$2\text{R}\cos x\cos y\cos z - \text{R}\cos(x+y+z),\ (x+y+z).$$

Similarly, the foot of the perpendicular from P′ on its Simson-line is

$$2\text{ R}\sin x\sin y\sin z + \text{R}\sin(x+y+z),\ \left(x+y+z-\frac{\pi}{2}\right).$$

The two Simson-lines will thus intersect at a point ω whose distanc from the initial line is

$$\{2\text{ R}\sin x\sin y\sin z + \text{R}\sin(x+y+z)\}\cos(x+y+z) -$$
$$\{2\text{ R}\cos x\cos y\cos z - \text{R}\cos(x+y+z)\}\sin(x+y+z).$$

Now
$$\cos(x+y+z) = \cos x\cos y\cos z\,(1-s_2)$$
$$\sin(x+y+z) = \cos x\cos y\cos z\,(s_1-s_3)$$
$$\sin(x+y)\sin(y+z)\sin(z+x) = \cos^2 x\cos^2 y\cos^2 z\,(s_1 s_2 - s_3);$$

where s_1, s_2, s_3 denote the sums of the products of tangents 1, 2, 3 at a time.

Thus, the distance of ω from PP′ is

$$2\text{ R}\cos^2 x\cos^2 y\cos^2 z\,\{s_3 + s_1 - s_3)(1-s_2) - (s_1 - s_3)\}$$
$$= -2\text{ R}\cos^2 x\cos^2 y\cos^2 z\,(s_1 s_2 - s_3)$$
$$= -2\text{ R}\sin(x+y)\sin(y+z)\sin(z+x)$$
$$= 2\text{ R}\cos\alpha\cos\beta\cos\gamma,$$

since the inclinations of the sides of ABC to PP′ are $\left(\frac{\pi}{2}+\alpha+\beta\right)$, &c.

Question 467.

(N. SANKARA AIYAR, M. A.) :—If O is the circumcentre of ABC and A′B′C′ the triangle whose sides bisect OA, OB, OC perpendicularly prove that AA′, BB′, CC′ are concurrent.

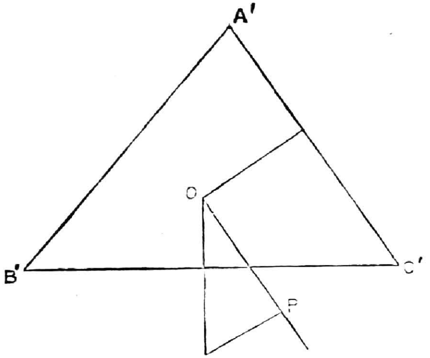
Additional Solution by Narasinga Rao, A.

It is evident that O is the incentre of A′B′C′.

Through O draw OP parallel to A′C′ and let AP be the ⊥ from A on this line.

Then the ⊥ from A on A′C′ $=r+$AP where r is the inradius of of A′B′C′.

But
$$r+\text{AP}=r\,(1+2\sin \text{AOP})$$
$$=r\,(1+2\cos \text{C}').$$



Hence the equation of AA′ referred to △ A′B′C′ is
$$\frac{\beta}{\gamma}=\frac{1+2\cos \text{C}'}{1+2\cos \text{B}'}.$$

Thus AA′, BB′, CC′ are concurrent at
$$\left(\frac{1}{1+2\cos \text{A}'},\ \frac{1}{1+2\cos \text{B}'},\ \frac{1}{1+2\cos \text{C}'}\right)$$

Question 472.

(SELECTED) :—Evaluate
$$\int_0^{\infty}\frac{\cos x}{(1+x)^2}dx,\ \int_0^{\infty}\frac{\cos 2x}{(1+x^2)^2}dx$$

Additional Solution by N. Sankara Aiyar, M. A.

$$\text{Let } J=\int_0^\infty \frac{\sin xy}{x(1+x^2)^2}dx$$

$$\therefore \quad \frac{d^4J}{dy^4}-2\frac{d^2J}{dy^2}+J=\int_0^\infty \frac{\sin xy}{x}dx=\frac{\pi}{2}.$$

$$\therefore \quad J=Aye^y+Bye^{-y}+Ce^y+De^y+\frac{\pi}{2}.$$

Now, J vanishes with $y=0$, and is infinite when $y=\infty$.

$$\therefore \quad J=Bye^{-y}+De^{-y}+\frac{\pi}{2}, \text{ whence } D=-\frac{\pi}{2}.$$

Differentiating

$$\frac{dJ}{dy}=\int_0^\infty \frac{\cos xy}{(1+x^2)^2}dx=-Bye^{-y}+(B-D)\,e^{-y}.$$

But
$$\frac{dJ}{dy}=\frac{\pi}{4} \text{ when } y=0.$$

$$\therefore \quad B-D=\frac{\pi}{4};\ \text{or } B=-\frac{\pi}{4}.$$

$$\therefore \quad \frac{dJ}{dy}=\frac{\pi}{4}ye^{-y}+\frac{\pi}{4}e^{-y}$$

$$\therefore \quad \int_0^\infty \frac{\cos x}{(1+x^2)^2}dx=\frac{\pi}{4}e^{-1}+\frac{\pi}{4}e^{-1}=\frac{\pi}{2}e^{-1}$$

$$\therefore \quad \int_0^\infty \frac{\cos 2x}{(1+x^2)^2}dx=\frac{\pi}{2}e^{-2}+\frac{\pi}{4}e^{-2}=\frac{3\pi}{8}e^{-2}$$

Question 475.

(K Appukuttan Erady, M.A.):—The space bounded by the coordinate planes and the surface $(x/a)^n+(y/b)^n+(z/c)^n=1$ is filled with an elastic fluid without weight. Prove that the pressures on the curved surface reduce to a single resultant whose line of action is

$$a(x-\lambda a)=b(y-\lambda b)=c(z-\lambda c),$$

where $\lambda=2\,\{\,\Gamma(2/n)\,\}^2\div 3\,\{\,\Gamma(1/n).\Gamma(3/n)\,\}$.

Solution by V. K. Aravamudan, B.A., R. Jagannathan and T. S. Krishna Rao.

As usual in such cases, project the surface on the three co-ordinate planes and find the resultant pressures along three mutually perpendicular directions.

Since the fluid is elastic and without weight

$$\frac{dp}{p}=0, \text{ or } p=\text{const.}=k(\text{say}),$$

First, projecting on the xy plane, the resolved pressure Z along ZO is $k\iint dx\,dy$, the integral being taken for all positive values of x and y given by the limiting equation $(x/a)^n+(y/b)^n=1$.

Hence its value is

$$k\,a\,b\frac{\left(\Gamma\frac{1}{n}\right)^2}{\Gamma\left(1+\frac{2}{n}\right)}$$

The co-ordinates of the point through which it acts are given by

$$\frac{\iint x\,dx\,dy}{\iint dx\,dy};\ \frac{\iint y\,dx\,dy}{\iint dx\,dy};\ 0.$$

Using Dirichlet's integrals, these are seen to be

$$\frac{a^2b\,\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{n}\right)\Gamma\left(1+\frac{2}{n}\right)}{ab\,\Gamma\left(\frac{1}{n}\right)^2\Gamma\left(1+\frac{3}{n}\right)},\quad \frac{a\,b^2\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{n}\right)\Gamma\left(1+\frac{2}{n}\right)}{ab\,\Gamma\left(\frac{1}{n}\right)^2\Gamma\left(1+\frac{3}{n}\right)},\ 0;$$

i.e.,
$$\frac{2}{3}\frac{\left(\Gamma\frac{2}{n}\right)^2}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{3}{n}\right)}a,\ \frac{2}{3}\frac{\left(\Gamma\frac{2}{n}\right)}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{3}{n}\right)}b,\ 0;$$

i.e.,
$$a\lambda,\ b\lambda,\ 0.$$

Similarly, the other components are

$X=\mu/a$, acting at $(o,\ b\lambda,\ c\lambda)$,
$Y=\mu/b$, acting at $(a\lambda,\ o,\ c\lambda)$,

where
$$\mu=\frac{k\,a\,b\,c\ \Gamma\left(\frac{1}{n}\right)^2}{\Gamma\left(1+\frac{2}{n}\right)}$$

Now, since each of these passes through $(a\lambda, b\lambda, c\lambda)$, the resultant is a **single** pressure whose direction cosines are proportional to x, y, z, *i.e.*, to $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

Thus its line of action is $a(x-a\lambda)=b(y-b\lambda)=c(z-c\lambda)$.

Question 482.

(A. NARASINGA RAO) :—If d_r be the number of divisors of integer r (unity excluded), shew that

$$L(d_1+d_2+\ldots d_n) \to n \log n, \text{ as } n \to \infty,$$

Solution by A. C. L. Wilkinson, M.A., F.R.A.S.

The following asymptotic formula is given by Dirichlet, *Journal de Liouville*, 1856, p. 359.

$$d_1+d_2+d_3+d_n=n \log n+(2C-1)\, n+\ldots\ldots$$

where C is Euler's constant. A proof will be found in Cesåro: *Calcolo Infinitesimale*, p. 96.

Question 483.

(V. V. S. NARAYAN):—Construct a triangle ABC being given the angle A, the side AB, and the distance of the middle point of AC from the symmedian point.

Solution by N. P. Pandya.

Let K be the symmedian point and E the midpoint of AC, and let EK$=p$ (a given length.)

By trilinear coordinates, E is the point $\left(\frac{\Delta}{a}, \text{o}, \frac{\Delta}{c}\right)$; K is the point

$$\frac{2a\Delta}{a^2+b^2+c^2}, \frac{2b\Delta}{a^2+b^2+c^2}, \frac{2c\Delta}{a^2+b^2+c^2}.$$

$$\therefore\ p^2=\frac{abc}{4\Delta^2}\left\{a \cos A\left(\frac{\Delta}{a}-\frac{2a\Delta}{a^2+b^2+c^2}\right)^2+b \cos B\left(\frac{-2b\Delta}{\Sigma a^2}\right)^2 +c \cos C\left(\frac{\Delta}{c}-\frac{2c\Delta}{a^2+b^2+c^2}\right)^2\right\}$$

$$=\frac{b^4}{4\,(b^2+c^2-bc \cos A)^2}(3c^2 \cos^2 A+b^2+c^2-4bc \cos A),$$

after sufficient reduction.

Thus, since p, c and A are known, b can be found from the above, and the required triangle is therefore determined.

Question 484.

(K. APPUKUTTAN ERADY M. A.):—If $\phi(xyz)\equiv(abcfgh)(x,y,z),^2$ prove that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\phi} \cos\ (px+qy+rz)\, dx\, dy\, dz=\frac{\pi^{\frac{3}{2}}}{\Delta^{\frac{1}{2}}}e^{-\lambda}.$$

where Δ is the discriminant of ϕ and $\lambda=\frac{(ABCFGH)(pqr)^2}{\Delta}$

Solution by R. Jagannathan.

Consider the conic $\phi=0$, and the straight line $px+qy+rz=0$. It is easy to see that by changing the triangle of reference to a self conjugate Δ having as side $Z=0$ the line $px+qy+rz=0$, ϕ transforms into $\alpha X^2+\beta Y^2+\gamma Z^2$ and $px+qy+rz$ into $2bZ$, say.

The integral in the question is therefore transformed into

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx \int_{-\infty}^{\infty} e^{-\beta y^2} dy \int_{-\infty}^{\infty} e^{-\gamma z^2} \cos.\ 2bz\ dz.$$

Now $$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-z^2} dz, \text{(putting } \sqrt{\alpha}\cdot x = z) = \frac{\sqrt{\pi}}{\sqrt{\alpha}}.$$

$$\int_{-\infty}^{\infty} e^{-\beta y^2} dy = \frac{\sqrt{\pi}}{\sqrt{\gamma}}, \text{ similarly.}$$

Also, $\int_{-\infty}^{\infty} e^{-\gamma z^2} \cos 2bz dz = \frac{\sqrt{\pi}}{\sqrt{\gamma}} . e^{-\frac{b^2}{\gamma}}$ (WILLIAMSON : *Integ. Calc. p*, 152.)

We have, further, the invariant relations

$\Delta = \alpha\beta\gamma$ and $Ap^2+Bq^2+Cr^2+2Fqr+2Gqr+2Hpq=4b^2\alpha\beta$.

$$\because \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(\alpha x^2+\beta y^2+\gamma z^2)} \cos 2bz\ dxdydz$$

$$= \frac{\pi^{3/2}}{\sqrt{\alpha\beta\gamma}} . e^{-\frac{b^2}{\gamma}} = \frac{\pi^{3/2}}{\sqrt{\alpha\beta\gamma}} . e^{-\frac{4b^2\alpha\beta}{4\alpha\beta\gamma}}$$

$$= \frac{\pi^{3/2}}{\Delta^{\frac{1}{2}}} . e^{-\lambda}, \text{ where } \lambda = \frac{(ABCFGH)\ (pqr)^2}{\Delta}.$$

From the invariant character of Δ and λ, it is at once clear that by retransforming from X ,Y, Z to (x, y, z) our original coordinates, we get the general theorem asked for

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\phi} \cos\ (px+qy+rz)\ dx\ dy\ dz$$

$$= \frac{\pi^{3/2} e^{-\lambda}}{\Delta^{\frac{1}{2}}}$$

Question 486.

(A. N. RAGHAVACHAR, M. A.) :—If $\alpha, \beta, \gamma, \delta$ be the roots of $x^4+px^3+qx^2+rx+s=0$, form the equation whose roots are

$$\frac{\beta\gamma(\alpha+\delta)-\alpha\delta(\beta+\gamma)}{\beta+\gamma-\alpha-\delta}, \text{ etc, etc.}$$

Solution by the Proposer.

Let $\rho=\dfrac{\beta\gamma(\alpha+\delta)-\alpha\delta(\beta+\gamma)}{\beta+\gamma-\alpha-\delta}$; then, by cross multiplication and tranposition, we have

$$\therefore \quad (\alpha+\delta)(\beta\gamma+\rho)=(\beta+\gamma)(\alpha\delta+\rho).$$

$$\therefore \quad \frac{\alpha+\delta}{\rho+\alpha\delta}=\frac{\beta+\gamma}{\rho+\beta\gamma}=-\frac{p}{2\rho+\alpha\delta+\beta\gamma};$$

$$=\frac{(\alpha+\delta)\beta\gamma+(\beta+\gamma)\alpha\delta}{\rho(\beta\gamma+\alpha\delta)+2\alpha\beta\gamma\delta}=-\frac{r}{\rho(\beta\gamma+\alpha\delta)+2s}.$$

Thus $\quad (\beta\gamma+\alpha\delta)(p\rho-r)=2(\rho r-ps) \qquad \ldots \qquad \ldots \quad (1)$

Again $\quad \rho^2=\dfrac{(\Sigma\alpha\beta\gamma)^2-4\alpha\beta\gamma\delta(\alpha+\delta)(\beta+\gamma)}{(\Sigma\alpha)^2-4(\alpha+\delta)(\beta+\gamma)}$

$$=\frac{r^2-4\alpha\beta\gamma\delta(\alpha+\delta)(\beta+\gamma)}{p^2-4(\alpha+\delta)(\beta+\gamma)}$$

$$=\frac{(r^2-4sq)(p\rho-r)+8s(\rho r-ps)}{(p^2-4q)(p\rho-\gamma)+8(\rho r-ps)}, \text{ from (1)}$$

In other words, ρ is a root of the equation

$$\rho^3(p^3-4pq+8r)-\rho^2(rp^2-4qr+8ps)+\rho(4pqs-pr^2-8rs)$$
$$+(r^3-4qrs+8ps^2)=0.$$

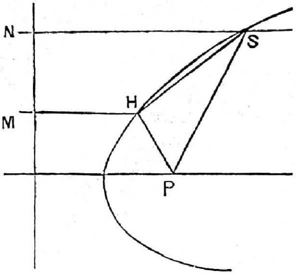
Question 487.

(N. P. PANDYA) :—If P be a point on a hyperbola whose foci are S and H, show that its asymptotes are parallel to the axes of the parabola passing through S, H and having P for their common focus.

Solutions (1) *by T. P. Trivedi, M.A., L.L.B.,* (2) *by R. Srinivasan, M.A., D. Krishna Murti, V. K. Aravamudan, B.A., and A. Narasinga Rao*; (3) *by N. Sankara Aiyar, M.A.*

(1) Drawing any one parabola through S and H having P for the focus, we see that the inclination of the axis of the parabola to SH being θ,

$$\cos\theta=\frac{\text{SN}-\text{HM}}{\text{SH}}=\frac{\text{SP}-\text{HP}}{\text{SH}}=\frac{2a}{2ae},$$



where e is the eccentricity of the hyperbola.

$$\therefore \qquad \cos\theta = \frac{1}{e} = \frac{a}{\sqrt{a^2+b^2}}. \qquad \therefore \tan\theta = \frac{b}{a}.$$

But the asymptotes of the hyperbola are inclined to SH on eithers side at an angle $\tan^{-1}(b/a)$; hence these asymptotes are parallel to the axe of the two parabolas which can be drawn with the given data.

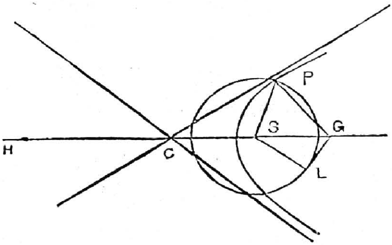
(2) Let PG be the normal at P. The directrix of the parabolas are the common tangents of circles described with centres S and H, and radii SP and HP.

Clearly G is the external centre of similitude of these circles.

Let GL be the tangent to the circle (S) at L.

We know that SG : SP $=e=$ SG : SL.

$$\therefore \qquad \angle LSG = \cos^{-1}\left(\frac{1}{e}\right)$$



Thus SL (which is parallel to the axis of one of the parabolas) is parallel to one asymptote.

Similarly, for the other.

(3) Let P be $a\sec\alpha$, $b\tan\alpha$. It. Then the parabolas will be

$$(a\sec\alpha - x)^2 + (b\tan\alpha - y)^2 = (x\cos\beta + y\sin\beta - p)^2.$$

Since these pass through S and H we get

$$a^2(\sec \alpha - e)^2 + b^2 \tan^2 \alpha = (ae \cos \beta - p)^2$$
$$a^2(\sec \alpha + e)^2 + b^2 \tan^2 \alpha = (ae \cos \beta + p)^2.$$

Subtracting

$$4\, a^2\, e \sec \alpha = 4\, ae\, p \cos \beta.$$

i.e., $a \sec \alpha = p \cos \beta$, or $p = a \sec \alpha \sec \beta$. Substituting this value of p, we get

$$a^2(\sec \alpha - e)^2 + b^2 \tan^2 \alpha = (ae \cos \beta - a \sec \alpha \sec \beta)^2,$$

i.e., $\sec^2\alpha - 2\, e \sec \alpha + e^2 + e^2 \tan^2\alpha - \tan^2\alpha$
$= e^2 \cos^2 \beta - 2\, e \sec \alpha + \sec^2\alpha \sec^2\beta$

i.e., $1 + e^2 \sec^2\alpha = e^2 \cos^2\beta + \sec^2\alpha \sec^2\beta^4$

i.e., $\sec^4\beta \sec^2\alpha - \sec^2\beta - e^2 \sec^2\beta \sec^2\alpha + e^2 = 0.$

i.e., $(\sec^2\beta - e^2)(\sec^2\alpha \sec^2\beta - 1) = 0.$

i.e., $\sec^2\beta = e^2$, the other solution being inadmissible.

i.e., $\sec \beta = \pm e.$

The axis which is perpendicular to the directrix $x \cos \beta + y \sin \beta - p = 0$, therefore, makes an angle $\sec^{-1}(\pm e)$ with the axes of reference and is therefore parallel to an asymptote of the given hyperbola.

Question 488.

(ZERO) :—Prove that the roots of the determinant equation of the nth degree

$$\begin{vmatrix} x, & 1, & 0, & 0, & 0 & \dots & \dots \\ 1, & x, & 1, & 0, & 0 & \dots & \dots \\ 0, & 1, & x, & 1, & 0 & \dots & \dots \\ 0, & 0, & 1, & x, & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0,$$

are $2 \cos \dfrac{k\pi}{n+1}$ $(k = 1, 2, 3 \dots n)$.

Additional Solutions (1) *by T. P. Trivedi M. A., L. L. B., and* (2) *by K. J. Sanjana M. A.*

(1) Let $f_n(x)$ represent the value of the determinant of the nth order, and let $F_n(x) = 0$ represent an equation whose roots are $2 \cos \dfrac{k\pi}{n+1}$ $(k = 1, 2 \dots n)$. We shall show that the expressions $F_n(x)$ and $f_n(x)$ are identical.

Expanding the determinant in terms of the elements of the first row, we get

$$f_n(x)=x\begin{vmatrix} x, & 1, & 0, & 0 & \dots & \dots \\ 1, & x, & 1, & 0 & \dots & \dots \\ 0, & 1, & x, & 1 & \dots & \dots \\ 0, & 0, & 1, & x, & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} -1\begin{vmatrix} 1, & 1, & 0, & 0 & \dots & \dots \\ 0, & x, & 1, & 0 & \dots & \dots \\ 0, & 1, & x, & 1 & \dots & \dots \\ 0, & 0, & 1, & x, & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$=xf_{n-1}(x)-f_{n-2}(x).$$

i.e., $$f_n(x)+f_{n-2}(x)=xf_{n-1}(x).$$

Again, if y represents $\cos\frac{k\pi}{n+1}+i\sin\frac{k\pi}{n+1}$, then $y^{2n+2}-1=0$.

Dividing this by y^2-1 and writing the equation as

$$y^n+y^{n-2}+y^{n-4}+\dots+\dots\dots\dots\dots y^{-n+2}+y^{-n}=0,$$

we see that $F_n(x)$ is obtained by putting $y+\frac{1}{y}=x$ in this, since $y+\frac{1}{y}=2\cos\frac{k\pi}{n+1}$.

Again, we have identically.

$$(y^n+y^{n-2}+y^{n-4}+\dots\dots y^{-n+2}+y^{-n})+(y^{n-2}+y^{n-4}+\dots+y^{-n+4}+y^{-n+2})$$
$$=(y+y^{-1})(y_u^{-1}+y^{n-3}+\dots y^{-1+3}+y^{-n+1})$$

On substituting $y+y^{-1}=x$, we have the same relation as before; *viz.*,

$$F_n(x)+F_{n-2}(x)=x\,F_{n-1}(x)$$

Now, taking the determinants of the first three orders, we get the expressions

$$x, \quad \begin{vmatrix} x, & 1 \\ 1, & x \end{vmatrix}=x^2-1, \quad \begin{vmatrix} x, & 1, & 0 \\ 1, & x, & 1 \\ 0, & 1, & x \end{vmatrix}=x^3-2x;$$

and we know that the roots of $x=0$, $x^2-1=0$ and $x^3-2x=0$ are $2\cos\frac{k\pi}{2}(k=1)$; $2\cos\frac{k\pi}{3}(k=1$ or $2)$ and $2\cos\frac{k\pi}{4}(k=1, 2$ or $3)$.

Hence we have the same solutions for $f_1(x)$ and $F_1(x)$; $f_2(x)$ and $F_2(x)$; $f_3(x)$ and $F_3(x)$; hence by using the above relations, we see that ultimately $f_n(x)$ and $F_n(x)$ are identical.

(2) Putting the determinant $=f_n(x)$, we can readily show that $f_n(x)=x\,f_{n-1}(x)-f_{n-2}(x)$, or

$$f_n(x)+f_{n-2}(x)=x\,f_{n-1}(x).$$

Now it is well known that $\frac{\sin n\theta}{\sin\theta}$ is a function of $2\cos\theta$ of degree $n-1$; putting $\xi=2\cos\theta$ and denoting the function by F, we have

$$F_n(\xi)+F_{n-2}(\xi)=\frac{\sin(n+1)\theta+\sin(n-1)\theta}{\sin\theta}=\frac{2\sin n\theta\cos\theta}{\sin\theta}$$
$$=\xi F_{n-1}(\xi);$$

so that three consecutive F's are connected in exactly the same way as three consecutive f's.

Also, $F_1(\xi)=0$ gives $\frac{\sin 2\theta}{\sin\theta}=0$, or $2\cos\theta=0$, *i.e.*, $\theta=\frac{1.\pi}{2}$;

$F_2(\xi)=0$, gives $\frac{\sin 3\theta}{\sin\theta}=0$, or $4\cos^2\theta-1=0$, *i.e.*, $\theta=\frac{(1,2)\pi}{3}$;

and $F_3(\xi)=0$ gives $\frac{\sin 4\theta}{\sin\theta}=0$ or $\cos 2\theta\cos\theta=0$, *i.e.*, $\theta=\frac{(1,2,3)\pi}{4}$.

The first three solutions of $f_n(x)=0$, give $x=0$, $x=\pm 1$, and $x\pm\sqrt{2}$, which are the doubled cosines of the angles found above. Hence, the first three solutions coinciding and formula of recurrence being the same, it is evident that all succeeding solutions of the equations $f_n(x)=0$, and $F_n(\xi)=0$, will coincide.

The solution of $F_n(\xi)=0$ is that of $\frac{\sin(n+1)\theta}{\sin\theta}=0$.

If $\sin(n+1)\theta=0$, we have $\theta=\frac{(0,1,2,\ldots,n)\pi}{n+1}$, and omitting θ whicd correspond to $\sin\theta$ in the denominator, we see that the n values of ξ, and therefore of x, are

$$2\cos\frac{k\pi}{n+1},\ [k=1,2,\ldots,n].$$

I find from some notes that this problem was proposed by Mr. TITS in *Mathesis*, but have no recollection when and on what lines the solution was given.

Question 490.

(R. SRINIVASAN, M.A.) :—Shew that

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{(2n+1)\Gamma(n+\frac{3}{2})}=\sum_{n=0}^{\infty}\left\{\frac{4}{\sqrt{\pi}}\frac{(-1)^n}{(2n+1)^2}\right\}$$

Solution (1) *by S. Ramanujan*, (2) *by S. Krishnaswami Aiyangar.*

(1) The question should be as printed above.

The left side $=\frac{2}{\sqrt{\pi}}\left\{\frac{1}{1^2}+\frac{2}{3^2}+\frac{2\cdot 4}{3\cdot 5^2}+\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7^2}+\ldots\right\}$

$$=\frac{2}{\sqrt{\pi}}\int_0^1\frac{\sin^{-1}x}{x\sqrt{1-x^2}}dx,$$

$$=\frac{2}{\sqrt{\pi}}\int_0^{\frac{\pi}{4}}\frac{\frac{\pi}{4}+\theta.\cos 2\theta}{\sin 2\theta\cos 2\theta}d\theta,\text{ (putting } x=\sin 2\theta)$$

$$=\frac{4}{\sqrt{\pi}}\int_0^1\frac{1}{y}\tan^{-1}y\,dy\text{ (put } y=\tan\theta)$$

$$=\frac{4}{\sqrt{\pi}}\left\{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\dots\right\}=\sum_0^\infty\left\{\frac{4}{\sqrt{\pi}}\frac{(-1)^n}{(2n+1)^2}\right\}.$$

(2) $$\sum_0^\infty\frac{\Gamma(n+1)}{(2n+1)\Gamma(n+\frac{3}{2})}=\frac{2}{\sqrt{\pi}}\sum_0^\infty\int_0^{\frac{\pi}{2}}\frac{\sin^{2n+1}\theta}{2n+1}d\theta$$

$$=\frac{2}{\sqrt{\pi}}\int_0^{\frac{\pi}{2}}\left\{\sin\theta+\frac{\sin^3\theta}{3}+\frac{\sin^5\theta}{5}+\dots\dots\right\}d\theta$$

$$=\frac{2}{\sqrt{\pi}}\int_0^{\frac{\pi}{2}}\left\{\tanh^{-1}(\sin\theta)\,d\theta.\right.$$

$$=\frac{2}{\sqrt{\pi}}\int_0^\infty x\text{ sech }x dx,\text{ changing }\sin\theta\text{ to }\tanh x.$$

$$=\frac{2}{\sqrt{\pi}}\int_0^\infty 2xe^{-x}(1+e^{-2x})^{-1}dx.$$

$$=\frac{4}{\sqrt{\pi}}\int_0^\infty\left\{xe^{-x}-xe^{-3x}+xe^{-5x}\dots\dots\right\}d$$

$$=\frac{4}{\sqrt{\pi}}\left\{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\dots\dots\dots+(-)^n\frac{1}{(2n+1)^2}+\dots\dots\right\}$$

$$=\sum_0^\infty\left\{\frac{4}{\sqrt{\pi}}(-)^n\frac{1}{(2n+1)^2}\right\}$$

Question 491.

(M. T. NARANIENGAR,) :—In any triangle prove that a circumconic passing through the ends of a diameter of the maximum inscribed ellipse touches the ellipse.

Solution by A. Narasinga Rao.

Project the triangle into an equilateral triangle. Then the maximum inscribed ellipse projects into the incircle of the equilateral triangle. Also the isogonal transformation of the incircle is the three-

cusped hypocycloid with the cusps at the vertices, and any line through the centre of the triangle transforms into a rectangular hyperbola through the vertices.

Hence the theorem reduces to :

If a rectangular hyperbola through the vertices of the triangle cuts the tricusp at P aud Q, then PQ touches the tricusp.

This property has been proved in the Journal. (*Vide*: Vol, V. p. 86.)

Question 492.

(K. V. Anantnarayan Sastri, B.A. :—Expand $\theta \tan\frac{\theta}{2}$ in powers of $\sin\theta$.

Solution (1) *by T. P. Trivedi, M.A., LL.B., and S. Ramanujan* (2) *by R. Vythynathaswamy.*

$$(1)\quad \theta\tan\frac{\theta}{2}=\frac{\theta\sin\theta}{1+\cos\theta}=\frac{x\sin^{-1}x}{1+\sqrt{1-x^2}},$$

$$=\frac{\sin^{-1}x(1-\sqrt{1-x^2})}{x},\ \text{if } x=\sin\theta,$$

Now $$\frac{\sin^{-1}x}{x}=1+\tfrac{1}{2}\cdot\frac{x^2}{3}+\frac{1\cdot3}{2\cdot4}\,\frac{x^4}{5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{x^6}{7}+\ldots$$

Again $$\frac{\sin^{-1}x}{\sqrt{1-x^2}}=x+\frac{2}{3}x^3+\frac{2\cdot4}{3\cdot5}x^5+\frac{2\cdot4\cdot6}{3\cdot5\cdot7}x^7+\ldots$$

$$\therefore\quad \frac{\sin^{-1}x\sqrt{1-x^2}}{x}=\frac{1}{x}\left(x+\frac{2}{3}x^3+\frac{2\cdot4}{2\cdot5}x^5+\frac{2\cdot4\cdot6}{3\cdot5\cdot7}x^7+\ldots.\right)(1-x^2)$$

$$=1-\frac{x^2}{3}-\frac{2}{3}\cdot\frac{x^4}{5}-\frac{2\cdot4x^6}{3\cdot5\,7}-\ldots$$

Hence

$$\frac{\sin^{-1}x(1-\sqrt{1-x^2})}{x}=\frac{x^2}{3}(1+\tfrac{1}{2})+\frac{x^4}{5}\left(\frac{2}{3}+\frac{1\cdot3}{2\cdot4}\right)$$
$$+\frac{x^6}{7}\left(\frac{2\cdot4}{3\cdot5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}+\ldots\ldots\right)$$

Replacing x by $\sin\theta$, the result is obtained.

(2) Let $t=\tan\frac{\theta}{2}$ and $s=\sin\theta$. Then

$$t^2s-2t+s=0.$$

$\therefore$ [Edwards *Diff. Cal.* Page 457, Ex. 8 (1).]

$$\therefore\quad t=\frac{s}{2}+\frac{s^3}{3^2}+\frac{4s^5}{2\cdot2^5}+\frac{5.6}{2\cdot3}\,\frac{s^7}{2^7}+\ldots\ldots$$

and $$\theta=s+\frac{1}{2\cdot3}\frac{s^3}{}+\frac{1\cdot3}{2\cdot4}\frac{s^5}{5}+\ldots\ldots$$

By multiplying the two series, we get the expansion as given.

QUESTIONS FOR SOLUTION.

508. (S. P. SINGARAVELU MODELIAR) :—If s_n stand for the sum of the reciprocals of the first n natural numbers, find the sum of the infinite series $s_1-\frac{1}{2}(\frac{1}{2})^2 s_2+\frac{1}{3}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 s_3-+\ldots\ldots\ldots$

509. (K. APPU KUTTAN ERRADY, M.A.) :—Forces act at the point (f, g, h) within the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, and are represented by the normals from the point to the surface. Show that the resultant acts along the lines whose direction cosines are proportional to

$$f\left(\frac{b^2}{a^2-b^2}+\frac{c^2}{a^2-c^2}-1\right), \text{ etc., etc.,}$$

and find its magnitude.

510. (S. P. SINGARAVELU MUDALIAR) :—Shew that the locus of the symmedian point of the triangle of maximum area inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{4}\cdot\left(\frac{a^2-b^2}{a^2+b^2}\right).$$

511. (S. KRISHNASWAMI AIYANGAR) :—If S and H are the foci of the maximum inscribed ellipse of the triangle of reference prove that

$$AS\cdot BS\cdot CS\cdot AH\cdot BH\cdot CH=\frac{a^2b^2c^2}{27}.$$

512. (A NARASINGA RAO) :—If $n\phi(n)+(n+2)\ \phi\ (n+2)=\phi\ (n+1)$, prove that

$$\text{(i)}\ \phi\ (1)x-\phi\ (3)\ x^3+\phi\ (5)\ x^5-\ldots\ldots=\phi\ (1)\sin\ (\tanh^{-1}x).$$

$$\text{(ii)}\ \phi\ (0)-\phi\ (2)\ x^2+\phi\ (4)\ x^4-\ldots\ldots=\phi\ (0)+\phi\ (1)(\cos\ \tanh^{-1}x+1).$$

Discuss the convergency of the series $\Sigma\phi\ (n)\ x^n$.

513. (PROFESSORS T. P. TRIVEDI AND K. J. SANJANA).—Find the sum of the series

$$1-\frac{h}{h-k+2}+\frac{h\ (h-1)}{(h-k+2)\ (h-k+3)}-\frac{h\ (h-1)\ (h-2)}{(h-k+2)\ (h-k+3)\ (h-k+4)}+\ldots,$$

where h is a positive integer and k any rational number.

514. (A. C. L. WILKINSON M.A., F.R., A.S.) :—All conics cutting a rectangular hyperbola orthogonally at all four points of intersection consist of (1) hyperbolas having the axes of the given hyperbola as

asymptotes, (2) all conics confocal with the given hyperbola, (3) two sets of ellipses each passing through two fixed points and whose centres line on one or other of the axes of the hyperbola and whose axes are parallel to the asymptotes of the hyperbola.

515. (A. C. L. Wilkinson M.A., F.R A.S.) :—ABCD is a quadrilateral. DQ, BP are any two parallel straight lines meeting AB, CD respectively in Q and P ; QL parallel to CD meets BC in L and PN parallel to AB meets AD in N ; prove that the middle points of AC, DL, BN are collinear.

516. (K. J. Sanjana, M. A.) :—TP, TQ tangents to a conic of centre C and focus F, cut the auxiliary circle in Y, Z, and FW is perpendicular to the chord of contact ; a tangent of the conic perpendicular to TC cuts FY, FZ, FW in y, z, w, respectively. Prove that $yw = wz$, and enunciate the corresponding theorem for the parabola.

517. (T. P. Trivedi M.A., L.L B.)—Is there any known method of integrating completely

$$\left(\frac{d^2y}{dx^2}\right)^2 \frac{d^5y}{dx^5} - \frac{d^2y}{dx^2}\frac{d^3y}{dx^3}\frac{d^4y}{dx^4} + \frac{40}{9}\left(\frac{d^3y}{dx^3}\right)^3 = 0,$$

which is the well-known differential equation of all conic sections ?

518. (R. Srinivasan) :—If the centroid of the triangle formed by the normals to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points whose eccentric angles are α, β, γ, lies on the major axis, shew that

$$3+4\Sigma\cos\alpha\Sigma\cos\beta\cos\gamma+2\Sigma\cos2\alpha+\Sigma\cos2(\alpha+\beta)=0.$$

519. (V. Anantaraman) :—Given the base and the vertical angle of a triangle find the locus of the centre of the circle passing through the three excentres.

520. (R. Vythynathaswamy) :—ABCD is a quadrilateral inscribed in a circle, centre O. Denoting the simson line of A *w. r. t.* BCD by a and so on, shew that, if a, b, c, d intersect OA, OB, OC, OD in P, Q, R, S and themselves cointersect at T, then P, Q, R, S, T, O are cyclic.

521. (K. V. Anantanarayana Sastry, B.A.) :—If V denote the entire volume of the figure bounded by $(x/a)^{\frac{16}{17}}+(y/b)^{\frac{16}{17}}+(z/c)^{\frac{16}{17}}=1$, and A the whole area of its trace on the xy-plane, prove that

$$V=\frac{17A.c}{19\cdot35}.B\left(\frac{1}{8},\frac{1}{16}\right).$$

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522. (V. V. S. Narayan) :—Given a parallelogram (drawn on paper) and an ungraduated straight edge, show how to trisect a given finite straight line drawn on the same paper.

523. (R. N. Apte, M.A., F.R.A.S.,) :—Find the value of

$$\iint x\,y\,z\left\{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2\right\}dx\,dy,$$

where $z=c\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}$, and the integration is over the positive quadrant of $x^2/a^2+y^2/b^2=1$.

524. (S. Ramanujan) :—Shew that

$$\text{(i)}\quad \sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{4\pi}{7}}+\sqrt[3]{\cos\frac{8\pi}{7}}=\sqrt[3]{\frac{5-3\sqrt[3]{7}}{2}}$$

$$\text{(ii)}\quad \sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}}=\sqrt[3]{\frac{3\sqrt[3]{9}-6}{2}}.$$

525. (S. Ramanujan) :—Shew how to find the square roots of surds of the form $\sqrt[3]{A}+\sqrt[3]{B}$, and hence prove that

$$\sqrt{\sqrt[3]{5}-\sqrt[3]{4}}=\frac{\sqrt[3]{2}+\sqrt[3]{20}-\sqrt[3]{25}}{3};$$

and

$$\sqrt{\sqrt[3]{28}-\sqrt[3]{27}}=\frac{\sqrt[3]{98}-\sqrt[3]{28}-1}{3}.$$

526. (S. Ramanujan) :—If n is any positive quantity shew that $\frac{1}{n}>\frac{1}{n+1}+\frac{1}{(n+2)^2}+\frac{3}{(n+3)^3}+\frac{4^2}{(n+4)^4}+\frac{5^3}{(n+5)^5}+\ldots$ Find the difference approximately when n is great.

Hence shew that

$$\frac{1}{1001}+\frac{1}{1002^2}+\frac{3}{1003^3}+\frac{4^2}{1004^4}+\frac{5^3}{1005^5}+\ldots<\frac{1}{1000} \text{ by } 10^{-440} \text{ nearly.}$$

527. (A. Narasinga Rao) :—A family of quartics touches each side of triangle ABC at A, B, C and passes through four given points. Prove that only two members of the family will touch the circumcircle of ABC.

528. (A. N. RAGHAVACHAR M.A.):—If (n, r) denote the sum of the products r together of the n natural numbers, find the value of

$$\frac{(2n-1,n)}{\lfloor 2n} - \frac{(2n-2,n)}{\lfloor 1 \lfloor 2n-1} + \frac{(2n-3,n)}{\lfloor 2 \lfloor 2n-2} + \cdots \frac{(-)^{n-1}(n,n)}{\lfloor n-1 \lfloor n+1}.$$

529. (N. P. PANDYA):—In a triangle ABC, D, E, F are the mid-points of BC, CA, AB. If AX, BY, CZ are perpendiculars on the sides of DEF, shew geometrically that DX, EY, FZ are concurrent at the symmedian point of ABC.

530. (N. SANKARA AIYAR M.A.):—Solve in integers

$$\frac{x^2+17}{y+9} = \text{a perfect square}.$$

531. (N. SANKARA AIYAR, M.A.):—Integrate

$$\int_0^\infty \frac{\sin xy}{x(1+x^2)^3} dx.$$